Optimization and strictly increasing transformation

Given the function $f(x, y) = 10x^{1/5}y^{4/5}$ subject to $7x + 4y = 70$

1. Find the value (x, y) that maximizes the problem.
2. Now consider the problem of maximizing the function $g(x, y) = \ln[f(x, y)]$, subject to the same constraint. What do you observe? Draw conclusions and justify.

Solution

1. We set up the Lagrangian:
$$L = 10x^{1/5}y^{4/5} + \lambda(70 - 7x - 4y)$$
We compute the first order conditions:
$$L'_x = 2x^{-4/5}y^{4/5} - \lambda 7 = 0$$
$$L'_y = 8x^{1/5}y^{-1/5} - \lambda 4 = 0$$
$$L'_\lambda = 70 - 7x - 4y = 0$$
Solving from the first two equations:
$$\frac{2}{7}x^{-4/5}y^{4/5} = \lambda$$
$$2x^{1/5}y^{-1/5} = \lambda$$
Equating them:
$$\frac{2}{7}x^{-4/5}y^{4/5} = 2x^{1/5}y^{-1/5}$$
$$\frac{y}{7} = x$$
Inserting into the last constraint:
$$70 - 7\frac{y}{7} - 4y = 0$$
$$70 - 5y = 0$$
$$y = 14$$
Inserting this into the x equation:
$$\frac{14}{7} = x$$
$$x = 2$$

2. $g(x, y) = \ln[f(x, y)] = \ln(10)+\ln(x^{1/5})+\ln(y^{4/5}) = \ln(10)+\frac{1}{5}\ln(x)+\frac{4}{5}\ln(y)$. We set up the Lagrangian:
$$L = \ln(10) + \frac{1}{5}\ln(x) + \frac{4}{5}\ln(y) + \lambda(70 - 7x - 4y)$$
We compute the first order conditions:
$$L'_x = \frac{1}{5x} - \lambda 7 = 0$$
$$L'_y = \frac{4}{5y} - \lambda 4 = 0$$
$$L'_\lambda = 70 - 7x - 4y = 0$$
Solving from the first two equations:
$$\frac{1}{35x} = \lambda$$
$$\frac{1}{5y} = \lambda$$
Equating them:
$$\frac{1}{35x} = \frac{1}{5y}$$

Solving for x:

$$\frac{y}{7} = x$$

Inserting into the last constraint:

$$70 - 7\frac{y}{7} - 4y = 0$$
$$70 - 5y = 0$$
$$y = 14$$

Inserting this into the x equation:

$$\frac{14}{7} = x$$
$$x = 2$$

Thus, we obtain the same result again. This is because taking the natural logarithm of a function is a strictly increasing transformation and therefore does not alter the maxima or minima.
Let's check the second order conditions by constructing the bordered Hessian. We can construct it either with the original function f or with the new function g. As the derivatives are simpler, I'll use the function g:

$$L''_{xx} = -\frac{1}{5x^{-2}}$$
$$L''_{yy} = -\frac{4}{5y^{-2}}$$
$$L''_{xy} = L''_{yx} = 0$$

Now the derivatives of the constraint which we'll call $h(x, y)$

$$h'_x = 7$$
$$h'_y = 4$$

$$\bar{H} = \begin{pmatrix} 0 & h'_x & h'_y \\ h'_x & L''_{xx} & L''_{xy} \\ h'_y & L''_{yx} & L''_{yy} \end{pmatrix} = \begin{pmatrix} 0 & 7 & 4 \\ 7 & -\frac{1}{5x^{-2}} & 0 \\ 4 & 0 & -\frac{4}{5y^{-2}} \end{pmatrix}$$

Evaluating at the critical point:

$$\begin{pmatrix} 0 & 7 & 4 \\ 7 & -\frac{1}{20} & 0 \\ 4 & 0 & -\frac{1}{245} \end{pmatrix}$$

The determinant is:

$$-7\begin{vmatrix} 7 & 4 \\ 0 & -\frac{1}{245} \end{vmatrix} + 4\begin{vmatrix} 7 & 4 \\ -\frac{1}{20} & 0 \end{vmatrix} = +\frac{49}{245} + \frac{16}{20} > 0$$

Since the determinant of the bordered Hessian is positive, we have a maximum.